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Conserved charges in the chiral three-state Potts model

G M T Watts
Department of Mathematics, King's College London, The Strand, London, WC2R 2LS, UK

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Abstract. We consider the perturbations of the three-state Potts conformal field theory introduced by Cardy as a description of the chiral three-state Potts model. By generalizing Zamolodchikov's counting argument and by explicit calculation we find new inhomogeneous conserved currents for this theory. We conjecture the existence of an infinite set of conserved currents of this form and discuss their relevance to the description of the chiral Potts models.

1. Introduction

The chiral three-state spin chain (see, e.g., [1, 2]) has the Hamiltonian

$$H = -\frac{2}{\sqrt{3}} \sum_j \left(\mathrm{e}^{-\mathrm{i}\varphi/3}\sigma_j + \mathrm{e}^{\mathrm{i}\varphi/3}\sigma_j^\dagger\right) + \lambda\left(\mathrm{e}^{-\mathrm{i}\phi/3}\Gamma_j\Gamma_{j+1}^\dagger + \mathrm{e}^{\mathrm{i}\phi/3}\Gamma_j^\dagger\Gamma_{j+1}\right) \qquad (1)$$

where j labels the sites and the matrices σ_j and Γ_j at each site are

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

with $\omega = \exp(2\pi\mathrm{i}/3)$. If $\cos\varphi = \lambda\cos\phi$ then the model is known to be integrable [2], and is self-dual for $\phi = \varphi$. The Hamiltonian of the 'standard' three-state Potts model is obtained from (1) by setting $\varphi = \phi = 0$, and this has a second-order phase transition at $\lambda = 1$ which is described by a conformal field theory with $c = \frac{4}{5}$ [3].

The standard model can be viewed as a perturbation of this conformal field theory by a particular field [3], known as the thermal perturbation (λ corresponding to temperature), and it is known that the resulting massive field theory is integrable with an infinite set of holomorphic and anti-holomorphic conserved quantities [4]. The existence of several of these conserved quantities can be shown by a simple counting argument due to Zamolodchikov [4, 5]. It is interesting to note here that the corresponding perturbation of the lattice model is probably not integrable in the usual sense.

In [6], Cardy considered the possibility that the full chiral Potts model can be viewed as a perturbation of this conformal field theory. He pointed out that half of the conserved quantities (the anti-holomorphic, say) of the thermal perturbation remain conserved when an extra, chiral, perturbation is added and identified this doubly perturbed model depending on two free parameters with the self-dual sub-sector of the general chiral three-state Potts model, but the relation between the integrable sub-sector of the chiral spin chain and this integrable field theory was unclear. However, it is still remarkable that such a perturbation by two relevant fields is integrable as the only double perturbations previously known to be integrable are the staircase models [7] which have one relevant and one irrelevant perturbation.

One question outstanding was whether there are any counterparts in the doubly perturbed theory to the holomorphic conserved quantities of the thermal perturbation. In this article we show that the double perturbation does have further conserved quantities which reduce to the 'missing' holomorphic conserved quantities when the 'extra' perturbation is removed, and conjecture that all the conserved quantities of the thermal perturbation can be extended in this way to conserved quantities of the doubly perturbed model.

Zamolodchikov's counting argument generalizes to the case of a double perturbation and proves the existence of two extra conserved quantities; we have checked the existence of two more explicitly, but at the moment a general proof is lacking.

The outline of this paper is as follows. We first recall the conformal field theory treatment of the Potts model, the conserved quantities for the various perturbations, and how the existence of several of these can be deduced by Zamolodchikov's counting argument.

We then consider the conserved quantities of the doubly perturbed model, give a counting argument and a few explicit examples of conserved quantities of this model.

Finally we speculate on the possible implications of these results for the doubly-perturbed and general perturbations of the conformal Potts model and the relation of the perturbed conformal field theories to the spin chain.

2. The conformal field theory of the three-state Potts model

The three-state Potts model was one of the first conformal field theories to be studied [3] and is both a minimal model of the Virasoro algebra and of the W_3 algebra [8], and hence the same field content admits two descriptions.

The Potts model has $c = \frac{4}{5}$ and has Virasoro primary fields of weights

$$(h, \overline{h}) = (0,0), \quad (3,0), \quad (0,3), \quad (3,3), \quad (\tfrac{2}{3}, \tfrac{2}{3}), \quad (\tfrac{1}{15}, \tfrac{1}{15}), \quad (\tfrac{2}{5}, \tfrac{2}{5}), \quad (\tfrac{2}{5}, \tfrac{7}{5}),$$
$$(\tfrac{7}{5}, \tfrac{2}{5}), \quad (\tfrac{7}{5}, \tfrac{7}{5}). \qquad (2)$$

The field with weights $(3, 0)$ is the holomorphic generator $W(z)$ of the W_3 algebra, and we take its commutation relations to be

$$[W_m\,,\ W_n] = \tfrac{13}{10\,800}\, m(m^2-1)(m^2-4)\delta_{m+n} + \tfrac{13}{720}\,(m-n)\,(2\,m^2 - mn + 2\,n^2 - 8)\, L_{m+n}$$
$$+ \tfrac{1}{3}\,(m-n)\,\Lambda_{m+n}. \qquad (3)$$

By allowing non-standard normalizations for the fields $(\frac{2}{5}, \frac{7}{5})$, $(\frac{7}{5}, \frac{2}{5})$, $(\frac{7}{5}, \frac{7}{5})$ we can identify them as descendants of the W-primary field $(\frac{2}{5}, \frac{2}{5})$ as follows:

$$\left|\tfrac{7}{5}, \tfrac{2}{5}\right\rangle = W_{-1}\left|\tfrac{2}{5}, \tfrac{2}{5}\right\rangle \qquad \left|\tfrac{2}{5}, \tfrac{7}{5}\right\rangle = \overline{W}_{-1}\left|\tfrac{2}{5}, \tfrac{2}{5}\right\rangle \qquad \left|\tfrac{7}{5}, \tfrac{7}{5}\right\rangle = W_{-1}\overline{W}_{-1}\left|\tfrac{2}{5}, \tfrac{2}{5}\right\rangle.$$

2.1. Integrable perturbations and conserved quantities

The most general action for a perturbed conformal theory is

$$S = S_0 + V \qquad V = \int \mathrm{d}^2 z\ \Phi(z, \overline{z})$$

where S_0 is the action of the conformal field theory and the potential V is given in terms of some local field $\Phi(z, \overline{z})$.

A field $U(z)$ which is holomorphic in the unperturbed theory may develop $\bar{z}$ dependence in the perturbed theory since its correlation functions are given as

$$\langle U(z,\bar{z})\cdots\rangle = \sum_n \frac{1}{n!}\langle U(z)V^n\cdots\rangle_0 \tag{4}$$

where $\langle\cdot\rangle_0$ is calculated in the unperturbed theory. To first order, the $\bar{z}$ dependence of (4) arises from the operator product expansion of U and v:

$$U(z)\,\Phi(w,\bar{w}) = \cdots + \frac{\chi(w,\bar{w})}{z-w} + \cdots. \tag{5}$$

Hence, to first order

$$\bar{\partial}\,U(z,\bar{z}) = \chi(z,\bar{z}) + \text{total } z\text{-derivatives}. \tag{6}$$

If $\chi(w,\bar{w})$ is a total z-derivative, i.e. if $|\chi\rangle = L_{-1}|\xi\rangle$ for some state $|\xi\rangle$, then $\oint \mathrm{d}z\, U$ remains conserved to first order in perturbation theory. By an abuse of notation we shall say that U is conserved if $\bar{\partial}U = \partial\xi$ for some ξ. It is sometimes possible to show that no higher-order corrections are possible and that in this way a quantity is conserved to all orders in the perturbed theory. The $\bar{w}$ dependence of Φ, χ and ξ is essentially irrelevant when checking the conservation of $\oint \mathrm{d}z\, U(z)$ to first order, and so we drop this dependence for the rest of this section. We shall keep the notation $\Phi_{h,\bar{h}}$ for fields with z and $\bar{z}$ dependence, and use ϕ_h or $\overline{\phi}_{\bar{h}}$ for the chiral dependence of such a field.

A large class of integrable perturbations of conformal field theories are affine Toda field theories, for which the existence of an infinite number of conserved quantities has been proven by Feigin and Frenkel [9]. Two perturbations of the Potts model can be interpreted as affine Toda field theories (ATFTs), namely the perturbations by the fields $\phi_{2/5}$ and $\phi_{7/5}$. (NB: the $\Phi_{7/5,7/5}$ perturbation is irrelevant, and consequently there may be higher-order corrections to the conservation equation, but we shall ignore the anti-holomorphic dependence of perturbations and other such issues.)

The $\phi_{7/5}$ perturbation corresponds to the $a_1^{(1)}$ ATFT, or Sine–Gordon, theory which has conserved currents $\tilde{U}_\Delta(z)$ of weights $\Delta = 2n$.

The $\phi_{2/5}$ perturbation corresponds both to the $a_2^{(2)}$ ATFT with conserved currents U_Δ polynomial in L of weights $\Delta = 6n, 6n{+}2$, and also to the $a_2^{(1)}$ theory with conserved currents polynomial in both L and W of weights $\Delta = 3n, 3n{+}2$. In this case the currents of even weight $6n, 6n{+}2$ are common to both theories and are independent of W whereas the currents of odd weight $6n+3, 6n+5$ are odd under $W \to -W$.

We give the first few states $U_\Delta = U_\Delta(0)|0\rangle$ and $\tilde{U}_\Delta = \tilde{U}_\Delta(0)|0\rangle$ below. (NB: these expressions are not unique, since they are only defined up to the addition of total derivatives and null states. We have chosen representatives which are reasonably compact).

2.2. Zamolodchikov's counting argument

In [5] Zamolodchikov showed how a simple counting argument can prove the existence of conserved quantities. We recall the method since some elements of it will be useful later.

Let us consider the simple case of a current $U_n(z)$ of conformal weight n which is a polynomial in $L(z)$ and its derivatives, and a perturbation by a Virasoro primary field $\phi_h(z)$. We have

$$\left[\oint U_n\,,\ \oint \phi_h\right] = \oint \psi \tag{7}$$

Table 1. Conserved currents of the $(\frac{2}{5})$ and $(\frac{7}{5})$ perturbations.

$U_2 = L_{-2}\,\lvert 0\rangle$
$U_3 = W_{-3}\,\lvert 0\rangle$
$U_5 = L_{-2}W_{-3}\,\lvert 0\rangle$
$U_6 = \left(L_{-2}L_{-2}L_{-2} + \frac{21}{10}\,L_{-3}L_{-3}\right)\lvert 0\rangle$
$U_8 = \left(L_{-2}L_{-2}L_{-2}L_{-2} - \frac{159}{25}\,L_{-3}L_{-3}L_{-2} - \frac{249}{25}\,L_{-4}L_{-2}L_{-2}\right)\lvert 0\rangle$
$U_9 = \left(L_{-2}L_{-2}L_{-2}W_{-3} + \frac{1071}{10}\,L_{-2}L_{-2}W_{-5} + \frac{16\,731}{125}\,L_{-3}W_{-6}\right)\lvert 0\rangle$
$\tilde{U}_2 = L_{-2}\,\lvert 0\rangle$
$\tilde{U}_4 = L_{-2}L_{-2}\,\lvert 0\rangle$
$\tilde{U}_6 = \left(L_{-2}L_{-2}L_{-2} - \frac{7}{30}\,L_{-3}L_{-3}\right)\lvert 0\rangle$
$\tilde{U}_8 = \left(L_{-2}L_{-2}L_{-2}L_{-2} - \frac{229}{375}\,L_{-3}L_{-3}L_{-2} + \frac{871}{375}\,L_{-4}L_{-2}L_{-2}\right)\lvert 0\rangle$
$\tilde{U}_{10} = \left(L_{-2}L_{-2}L_{-2}L_{-2}L_{-2} + \frac{3821}{225}\,L_{-4}L_{-2}L_{-2}L_{-2} + \frac{657}{50}\,L_{-3}L_{-3}L_{-2}L_{-2} - \frac{99}{10}\,L_{-4}L_{-3}L_{-3}\right)\lvert 0\rangle$

Table 2. Characters of quasi-primary states in the Potts model.

$\tilde{\chi}_0 = 1 + q^2 + q^4 + 2\,q^6 + 3\,q^8 + q^9 + 4\,q^{10} + \ldots$
$\tilde{\chi}_3 = 1 + q^2 + q^3 + q^4 + q^5 + 3\,q^6 + 2\,q^7 + 4\,q^8 + 4\,q^9 + 6\,q^{10} + \ldots$
$\tilde{\chi}_{2/5} = 1 + q^3 + q^4 + q^5 + 2\,q^6 + 2\,q^7 + 3\,q^8 + 4\,q^9 + 5\,q^{10} + \ldots$
$\tilde{\chi}_{7/5} = 1 + q^2 + 2\,q^4 + q^5 + 3\,q^6 + 2\,q^7 + 5\,q^8 + 4\,q^9 + 7\,q^{10} + \ldots$

where $\lvert\psi\rangle = (U_n)_{-n+1}\,\lvert\phi_h\rangle$. If the dimension d_n^0 of the space of non-trivial integrals $\oint U_n$ is greater than the dimension d_{n-1}^h of non-derivative descendants of $\lvert\Phi\rangle$ at level $n-1$, then the existence of $d_n^0 - d_{n-1}^h$ conserved currents is guaranteed. The first few conserved currents of the $(\frac{2}{5})$ and $(\frac{7}{5})$ perturbations are given in table 1.

If we define the modified character of a Virasoro highest-weight representation of weight h by

$$\chi_h = \mathrm{Tr}\,q^{h-L_0} \tag{8}$$

then the characters $\tilde{\chi}_h$ of non-derivative, or quasi-primary, states are given by

$$\tilde{\chi}_0 = \sum d_n^0 q^n = (1-q)\,\chi_0 + q \qquad \tilde{\chi}_h = \sum d_n^h q^n = (1-q)\,\chi_h \quad (h \neq 0, -1\ldots). \tag{9}$$

Applying this to the Potts model, we find (to order q^{10}) the characters given in table 2.

Now, examining $\tilde{\chi}_0 - q\,\tilde{\chi}_{2/5}$ we can infer the existence of U_2, U_6, U_8 (and, expanding further, U_{12}), and examining $\tilde{\chi}_0 - q\,\tilde{\chi}_{7/5}$, of $\tilde{U}_2, \tilde{U}_4, \tilde{U}_6$ and $\tilde{U}_8$.

This method may be adapted to deduce the existence of the conserved quantities which are linear in W [4]. Since these currents are all Virasoro descendents of $W(z)$, the number of such quasi-primary fields of weight n is given by d_{n-3}^3. Similarly, since the operator product of a Virasoro descendant of $W(z)$ with $\phi_{2/5}$ is a Virasoro descendant of $\phi_{7/5}$, the number of quasi-primary fields of weight $(\frac{2}{5}+n-1)$ which may occur on the right-hand side of (7) is given by $d_{n-2}^{7/5}$. Consequently, to verify the existence of a conserved current of this form we need to check $q^3\,\tilde{\chi}_3 - q^2\,\tilde{\chi}_{7/5}$, and find that in this way U_3, U_5 and U_9 are guaranteed to exist.

3. The general perturbation of the three-state Potts model

As seen earlier, the general three-state Potts chain has three parameters and Cardy proposed that this corresponds to the action

$$S_0 + \int \mathrm{d}^2 z\,\left(\tau\,\Phi_{2/5,2/5} + \delta\,\Phi_{7/5,2/5} + \overline{\delta}\,\Phi_{2/5,7/5}\right). \tag{10}$$

The standard thermal perturbation of the three-state Potts model is given by $\delta = \overline{\delta} = 0$ and is integrable with the conserved currents $U_3, \ldots$. In [6] Cardy showed that the more general model with $\overline{\delta} = 0$ is also integrable by the following argument. Since the anti-holomorphic dependence of both fields $\Phi_{2/5,2/5}$ and $\Phi_{7/5,2/5}$ is the same, i.e. $\overline{\phi}_{2/5}(\overline{z})$, all the anti-holomorphic conserved currents of the thermal perturbation will remain conserved for this double perturbation. However, a quick glance at table 1 shows that there are no non-trivial holomorphic conserved currents common to both the $\phi_{2/5}$ and $\phi_{7/5}$ perturbations, and so it is not clear what will happen to the holomorphic conserved currents of the thermal perturbation when the perturbation δ is turned on.

However, it is important to note that the action (10) no longer preserves rotational, or Lorentz, invariance, and hence conserved currents need not have a well defined spin. For example, we can consider currents of the form

$$T_n = T_{(n,0)} + \frac{\delta}{\tau}\,T_{(n,1)} \tag{11}$$

where $T_{(n,0)}$ and $T_{(n,1)}$ are some conformal fields of weights n and $n+1$, respectively. Such a current will be conserved for the doubly perturbed ($\overline{\delta} = 0$) action if the following three equations hold:

$$\left[\oint T_{(n,0)}\,,\ \oint \phi_{2/5}\right] = 0 \tag{12}$$

$$\left[\oint T_{(n,1)}\,,\ \oint \phi_{7/5}\right] = 0 \tag{13}$$

$$\left[\oint T_{(n,0)}\,,\ \oint \phi_{7/5}\right] + \left[\oint T_{(n,1)}\,,\ \oint \phi_{2/5}\right] = 0. \tag{14}$$

Equations (12) and (13) imply that $T_{(n,0)} = \alpha\,U_n$ and $T_{(n,1)} = \beta\,\tilde{U}_{n+1}$ for some α, β. The extra requirement (14) can be ensured by a modification of Zamolodchikov's argument. In this case, if there is only a single quasi-primary descendent of $\phi_{2/5}$ at level n then both terms on the right-hand side of (14) must be proportional, and hence cancel for some choice of α/β. Examining $\tilde{\chi}_{2/5}$, we see that this does indeed happen for $n = 3$ and $n = 5$. As a result we have proven the existence of holomorphic conserved currents in Cardy's model.

The next possible value of n for which there might be a conserved current of the form (11) is $n = 8$, but explicit calculation shows that this trick will not work. However, we can instead extend the ansatz to include three terms

$$T_n = T_{(n,0)} + \frac{\delta}{\tau}\,T_{(n,1)} + \left(\frac{\delta}{\tau}\right)^2 T_{(n,2)} \tag{15}$$

Table 3. Conserved currents of the $(\frac{2}{5})$ plus $(\frac{7}{5})$ perturbation.

$$T_2 = U_2$$
$$T_3 = U_3 - \frac{7}{9}\tilde{U}_4$$
$$T_5 = U_5 - \frac{91}{180}\tilde{U}_6$$
$$T_6 = U_6 - \frac{147}{275}\,(12\,L_{-4}W_{-3} - 15\,L_{-3}W_{-4} + 10\,L_{-2}L_{-2}W_{-3})\,|0\rangle + \frac{175}{66}\,\tilde{U}_8$$
$$T_8 = U_8 - \frac{4\,837\,476}{1\,322\,035}\,(L_{-2}L_{-2}L_{-2}W_{-3} + \ldots)\,|0\rangle + \frac{343}{387}\,\tilde{U}_{10}$$

where now $T_{(n,0)} = \alpha\, U_n$, $T_{(n,2)} = \beta\, \tilde{U}_{n+2}$ and we have the non-trivial requirement that

$$\left[\oint T_{(n,0)}\,,\oint \phi_{7/5}\right] + \left[\oint T_{(n,1)}\,,\oint \phi_{2/5}\right] = \left[\oint T_{(n,1)}\,,\oint \phi_{7/5}\right] + \left[\oint T_{(n,2)}\,,\oint \phi_{2/5}\right] = 0. \tag{16}$$

We have verified that there are conserved currents of this form for $n = 6, 8$ by explicit calculation. We include these with the two previous conserved currents in table 3, in which we again give the states $T_n = T_n(0)|0\rangle$ and have set $\delta/\tau = 1$ for simplicity.

Finally we should remark that (with $\overline{\delta} = 0$) there are no further corrections to the conservation equation for T_n on dimensional grounds, and so this result should be exact to all orders in perturbation theory.

4. Remarks and conclusions

We have shown by counting arguments and explicit calculation that the double perturbation of the three-state Potts model considered by Cardy has extra conserved currents interpolating those known for the two constituent perturbing fields.

Although we have only constructed four conserved currents, an appealing pattern has appeared which suggests that there are conserved currents T_n for all $n = 0, 2 \bmod 3$, polynomial in $x = \delta/\tau$, and interpolating the conserved currents of the $\phi_{2,5}$ and $\phi_{7/5}$ perturbations:

$$\begin{aligned} T_{3n} &= U_{3n} + \ldots + x^n\,\beta_n\,\tilde{U}_{4n} \\ T_{3n+2} &= U_{3n+2} + \ldots + x^n\,\beta_n'\,\tilde{U}_{4n+2}. \end{aligned} \tag{17}$$

It is interesting to note that the conserved quantities in table 3 remain formally conserved to first order for the even more general action

$$S_0 + \int \mathrm{d}^2 z \left(\tau\,\Phi_{2/5,2/5} + \delta\,\Phi_{7/5,2/5} + \overline{\delta}\,\Phi_{2/5,7/5} + \left(\frac{\delta\overline{\delta}}{\tau}\right)\Phi_{7/5,7/5}\right). \tag{18}$$

This again relies on ignoring the $\overline{z}$ dependence of the perturbing fields, in which case we can formally factorize the potential as

$$\tau\left(\phi_{2/5} + \frac{\delta}{\tau}\,\phi_{7/5}\right)\left(\overline{\phi}_{2/5} + \frac{\overline{\delta}}{\tau}\,\overline{\phi}_{7/5}\right)$$

and it is clear that the new currents T_n are conserved for (18), as are similar currents $\overline{T}_n$ constructed from the anti-holomorphic algebra. This is not sufficient to conclude that this potential is integrable—the first order in perturbation theory is no longer exact since

$(\delta^3\bar{\delta}^3/\tau)$ is dimensionless and there are possible corrections to the conservation equation at arbitrarily high orders in perturbation theory. Furthermore, the explicit presence of an irrelevant field in the action spoils the property that the UV limit of the perturbed model is simply the conformal field theory.

An interesting point to notice is that the results of [10] indicate that $\phi = \varphi = \pi/2, 0.901\ldots < \lambda < 1.1095\ldots$ has massless modes described by a parity violating theory with $c = \bar{c} = 1$. It is believed that these values are continuously connected to the conformal point $\phi = \varphi = 0, \lambda = 1$ through massless theories, but it is hard to see how they can be reached by perturbation of the conformal three-state Potts, since the central charge of the conformal three-state Potts model is $\frac{4}{5}$ and one might expect central charge to decrease along renormalization group flows†. Perhaps the presence of an irrelevant field in the perturbation signals that it is a perturbation from a model with larger c, as happens, e.g., for the Virasoro minimal models where the irrelevant $\phi_{3,1}$ perturbation of the M_p model corresponds to the IR limit of the $\phi_{1,3}$ perturbation of the M_{p+1} model. However, one should remember that the true state of lowest energy has non-zero momentum and that it may be very hard to relate the exact results to those obtained in the perturbed conformal models.

It is also interesting to note that Cardy finds a different two-dimensional subspace of the space of coupling constants to be integrable ($\bar{\delta} = 0$) to that which appears to be the case from the transfer matrix approach ($\tau \sim \delta\bar{\delta}$) suggesting that it might be possible that both results are correct, consistent with the action (18) being integrable for all values of the coupling constants. Although the spin chain is not believed to be integrable for all values [11], it is possible that the scaling limit of the spin chain only differs from an integrable model by irrelevant operators, which, while breaking the exact integrability of the spin chain would give an integrable model in the IR. However, as Cardy notes, it is also possible that there is no relation between lattice integrability and the integrability of perturbed conformal models.

We should like to mention that there are well-known models which contain dimensionless parameters and which are believed to be integrable, for example the staircase models [7]. These are double perturbations of a conformal field theory by a relevant and an irrelevant operator which share the same conserved currents (to first order). While it is not possible to check integrability by exhibiting conserved currents exactly for the reason that there are dimensionless parameters, they do appear to share many properties of integrable models.

Finally, we should like to discuss possible generalizations of these results. Cardy's argument is sufficient to show that given any W-algebra and an integrable perturbation Φ then the anti-holomorphic conserved quantities for Φ remain conserved for any perturbation of the form $W_{-1}\Phi$. By contrast, to be able to apply our method to find holomorphic conserved currents for such a perturbation, it is also necessary that $W_{-1}\Phi$ is an integrable perturbation for some subalgebra of the W-algebra. In the model treated the original perturbation is $\Phi_{2/5}$ and $W_{-1}\Phi_{2/5}$ is an integrable perturbation for the Virasoro subalgebra of the W_3 algebra. However, it is easy to check that there are no other rational models of the W_3 algebra for which Φ is an integrable perturbation and $W_{-1}\Phi$ is an integrable perturbation of the Virasoro subalgebra.

The next obvious possible generalization is the Z_n chiral Potts models. These are described by a spin chain Hamiltonian given in terms of $(n \times n)$ matrices σ, Γ, and again dependent on three parameters λ, ϕ, φ. The point $\lambda = 1, \phi = \varphi = 0$ is now described by a $c = c_n = 2(n-1)/(n+2)$ conformal field theory which can be variously understood as

† Strictly speaking, the c-theorem which states that c is non-increasing along renormalization group flows only applies to Lorentz-invariant theories, but as c is a measure of the number of massless degrees of freedom, it is hard to see how it can actually increase when a theory is perturbed.

Table 4. $W_n^{(\mathrm{orb})}$ and related affine algebras.

n mod 4	$W_n^{(\mathrm{orb})}$	g_1	g_2	g_3	g_4
0	$WD_{n/4}$	$d_{n/4}^{(1)}$	$d_{n/4}^{(1)}$	$b_{n/4}^{(1)}$	$a_{(n-2)/2}^{(2)}$
2	$WB(0,(n-2)/4)$	$b_{(n-2)/4}^{(1)}$	$b_{(n-2)/4}^{(1)}$	$a^{(4)}(0,(n-2)/2)$	$b^{(1)}(0,(n-2)/4)$
1, 3	$WB_{(n-1)/2}$	$b_{(n-1)/2}^{(1)}$	$c_{(n-1)/2}^{(1)}$	$a_{(n-1)}^{(2)}$	$c_{(n-1)/2}^{(1)}$

the Z_n parafermion model [12], the first unitary minimal model of the W_n algebra [14] or a model with a $W(2,3,4,5)$ chiral algebra [16]. Several perturbations of this conformal field theory were studied in [13], and the thermal perturbation is given by a field of weight $h = h_n = 2/(n+2)$; Cardy has proposed that the chiral perturbation corresponds to a level-1 W-descendent of this field which, due to the null vectors in the representation $|h\rangle$, is probably of the form $W_{-1}^{(3)}|h\rangle$ (which is also the field ϕ of [13]). Given the results above, it is natural to suppose that this is itself an integrable perturbation for the subalgebra of the W-algebra which is invariant under the automorphism which sends the odd-spin generators $W \to -W$. The full W-algebra and its orbifold have been studied in some detail [15, 16], and it is believed that for $c = c_n$ the orbifold algebra is of the form $W(2,4,6,8)$. While it is not yet possible to study the representations of this algebra directly, there is some evidence that for each n it is in turn a truncation of some particular 'Casimir' W-algebra, which we denote by $W_n^{(\mathrm{orb})}$. This identification depends on (n mod 4) as given in table 4.

The identification of $W_n^{(\mathrm{orb})}$ for n even was made by Fateev in [13]. The evidence is that in each case the self-coupling of the spin 4 field as given in [16, 17] is the same in W_n and $W_n^{(\mathrm{orb})}$, that c_n is a minimal model $W_n^{(\mathrm{orb})}$, and that in each case $h_n, h_n{+}1$ and 3 are minimal model representations of $W_n^{(\mathrm{orb})}$ at $c{=}c_n$, as we detail below.

For each algebra $W_n^{(\mathrm{orb})}$ in table 4, the allowed representations at $c{=}c_n$ are labelled $h_{\lambda,\mu}$ where λ, μ are integrable weights of the two affine algebra g_1 and g_2 at levels 2 and 3 respectively. Then in each case we find amongst the allowed representations the values $h_{0,\Lambda_1} = 2/(n+2) = h_n$, $h_{0,2\Lambda_1} = (n+4)/(n+2) = h_n+1$ and $h_{0,3\Lambda_1} = 3$, where Λ_1 is the first fundamental weight of g_2.

These three equations suggest strongly that at $c = c_n$, the algebra $W_n^{(\mathrm{orb})}$ can be augmented by a representation of weight 3 to give the full W_n algebra; that the thermal perturbation is given by the field $(0, \Lambda_1)$, corresponding to the g_3 ATFT; and that there is a W_n descendent of this field at level 1 which is itself a highest weight of the $W_n^{(\mathrm{orb})}$ algebra of weight $h_n{+}1$ of type $(0, 2\Lambda_1)$ corresponding to the g_4 ATFT (see table 4 for g_3 and g_4). Hence both these perturbations are integrable, corresponding as they do to ATFTs.

Thus we suggest that the whole procedure in this article will also carry through for the Z_n chiral Potts models. The thermal perturbation corresponds to the $a_{n-1}^{(1)}$ ATFT with conserved currents of spins $2, \ldots, n$ mod n, and the descendent at level 1 to the g_4 ATFT with conserved currents of all even spins, and we expect conserved currents for the double perturbation interpolating these.

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References

[1] Howes S, Kadanoff L P and Den Nijs M 1983 Quantum model for commensurate–incommensurate transitions *Nucl. Phys.* B **215** 169–208
von Gehlen G and Rittenberg V 1985 Z_n symmetric quantum chains with an infinite set of conserved charges and Z_n zero modes *Nucl. Phys.* B **257** [FS14] 351–70
[2] Au Yang H, McCoy B M, Perk J H H, Tang Sh and Yan M L 1987 Commuting transfer matrices in the chiral Potts model and solutions of star–triangle equations with genus > 1 *Phys. Lett.* **123A** 219–23
[3] Dotsenko V S 1984 Critical behaviour and associated conformal algebra of the Z_3 Potts model *Nucl. Phys.* B **235** 54–74
[4] Zamolodchikov A B 1988 Integrals of motion in scaling 3-state Potts model field theory *Int. J. Mod. Phys.* A **3** 743–50
[5] Zamolodchikov A B 1989 Integrable field theory from conformal field theory *Adv. Stud. Pure Math.* **19** 641–74
[6] Cardy J L 1993 Critical exponents of the chiral Potts model from conformal field theory *Nucl. Phys.* B **389** 577
[7] Zamolodchikov Al B 1991 Resonance factorized scattering and roaming trajectories, *Ecole Normale Superieure preprint* ENS-LPS-335
Martins M J 1992 Renormalization group trajectories from resonance factorized S matrices *Phys. Rev. Lett.* **69** 2461–4
[8] Zamolodchikov A B 1985 Infinite additional symmetries in two-dimensional conformal quantum field theory *Theor. Math. Phys.* **65** 347–359
[9] Feigin B L and Frenkel E 1995 Integrals of motion and quantum groups *Proc. CIME Summer School on Integrable Systems and Quantum Groups (Lecture Notes in Mathematics 1620)* (Berlin: Springer)
[10] Albertini G and McCoy B M 1991 Correlation functions of the chiral Potts chain from conformal field theory and finite-size corrections *Nucl. Phys.* B **350** 745–88
[11] McCoy B M and Orrick W P 1996 Analyticity and integrability in the chiral Potts model *J. Stat. Phys.* **83** 839
[12] Zamolodchikov A B and Fateev V A 1985 Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems *Sov. Phys.–JETP* **62** 215–25
[13] Fateev V A 1991 Integrable deformations in Z_N-symmetrical models of the conformal quantum field theory *Int. J. Mod. Phys.* A **6** 2109–32
[14] Fateev V A and Luk'yanov S L 1988 The models of two-dimensional conformal quantum field theory with Z_N symmetry *Int. J. Mod. Phys.* A **3** 507
[15] de Boer J, Feher L and Honecker A 1993 A class of W Algebras with infinitely generated classical limit *Nucl. Phys.* B **420** 409–46
[16] Blumenhagen R, Eholzer W, Honecker A, Hornfeck K and Hübel R 1994 Unifying W-algebras *Phys. Lett.* **332B** 51
Blumenhagen R, Eholzer W, Honecker A, Hornfeck K and Hübel R 1995 Coset Realization of unifying W-algebras *Int. J. Mod. Phys.* A **10** 2367
[17] Hornfeck K 1994 Classification of structure constants for W-algebras from highest weights *Nucl. Phys.* B **B411** 307–20
[18] Frenkel E V, Kac V and Wakimoto M 1992 Characters and fusion rules for W-algebras via quantised Drinfel'd–Sokolov reduction *Commun. Math. Phys.* **147** 295–328